

STRATIFICATION OF LOCAL MODULI SPACES OF HIRZEBRUCH MANIFOLDS

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In this note we deal with the extension problem for holomorphic vector fields using an elementary method based on Čech cohomology groups. As an application, we study the structure of local moduli spaces of Hirzebruch manifolds.

Section 1 contains preliminary considerations on the extension problem for holomorphic vector fields in fiber manifolds, and in Section 2 we deduce explicit formulae for lower obstructions to extending a given vector field. In Section 3 we make some remarks on "discrete" fiber manifolds. We apply these theories, in Section 4, to the investigation of complete families of Hirzebruch manifolds, especially to the determination of the general fibers of these families. By the same method we can also determine the general fibers of the complete families of ruled surfaces constructed in [11].

§1. Extension problem for vector fields based on Čech cohomology groups

In this section, we make preliminary considerations on the extension problem for holomorphic vector fields. The method is similar to that in [5], where the extension problem for holomorphic mappings is treated.

Let V denote a compact complex manifold of complex dimension n . We consider a deformation $\mathcal{V} \xrightarrow{\tilde{\omega}} M$ of V , i.e., \mathcal{V} and M are connected complex manifolds, $\tilde{\omega}$ is a proper holomorphic mapping whose Jacobian is everywhere of maximal rank and, for some point $0 \in M$, $V_0 = \tilde{\omega}^{-1}(0)$ is isomorphic to V . Moreover let Θ denote the sheaf over V of germs of holomorphic vector fields. Then we have the infinitesimal deformation map $\rho: T_0(M) \rightarrow H^1(V, \Theta)$ (see [4]), where $T_0(M)$ denotes the complex tangent space of M at 0 . Given a vector field $\eta \in H^0(V, \Theta)$, we shall be concerned with the problem of determining the directions of the parameter space M to which η is extendible. Hence we begin by treating one-parameter families. Moreover we consider only "small" deformations of V . Thus we assume $M = D = \{t \in \mathbb{C} \mid |t| < 1\}$ is a disk.

The fiber manifold \mathcal{V} is covered by a finite number of coordinate neigh-

borhoods \mathscr{U}_j with local coordinates $(z_j, t) = (z_j^1, z_j^2, \dots, z_j^n, t)$, such that $\tilde{\omega}(z_j, t) = t$ and $\mathscr{U}_j = \{(z_j, t) \mid \max_\alpha |z_j^\alpha| < 1, |t| < 1\}$. Points $(z_j, t) \in \mathscr{U}_j$ and $(z_k, t) \in \mathscr{U}_k$ are identified if and only if

$$z_j = f_{jk}(z_k, t),$$

where $f_{jk}(z_k, t)$ is a vector-valued holomorphic function of (z_k, t) defined on $\mathscr{U}_j \cap \mathscr{U}_k$. Let \mathbf{T}_t denote the holomorphic tangent bundle of $V_t = \tilde{\omega}^{-1}(t)$. Then the vector bundle \mathbf{T}_t is defined by the 1-cocycle $\{T_{jk}(z_k, t)\}_j$, where $T_{jk}(z_k, t)$ is the $n \times n$ matrix:

$$T_{jk}(z_k, t) = \left(\frac{\partial f_{jk}^\alpha(z_k, t)}{\partial z_k^\beta} \right)_{\substack{\alpha=1,2,\dots,n, \\ \beta=1,2,\dots,n,}}$$

The sheaf Θ_t is given by $\Theta_t = \mathcal{O}_{V_t}(\mathbf{T}_t)$, where \mathcal{O}_{V_t} denotes the structure sheaf of V_t . We write $\Theta_0 = \Theta$.

Convention. For a function $f(t)$ of t , we denote $f(0)$ simply by f .

The extension problem for a given vector field $\eta \in H^0(V, \Theta)$ consists in finding $\eta_t \in H^0(V_t, \Theta_t)$ which depends holomorphically on t . To say this precisely, represent η by a system $\{\eta_j(z_j)\}$ of vector-valued holomorphic functions $\eta_j(z_j)$ on $U_j = \mathscr{U}_j \cap V$ satisfying $\eta_j(z_j) = T_{jk}(z_k)\eta_k(z_k)$, η is said to be *extendible* or *stable* in the fiber manifold $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ if there exists a vector-valued holomorphic function $\eta_j(z_j, t)$ on each \mathscr{U}_j such that $\eta_j(z_j, 0) = \eta_j(z_j)$ and that

$$(1) \quad \eta_j(f_{jk}(z_k, t), t) = T_{jk}(z_k, t)\eta_k(z_k, t)$$

whenever $(z_k, t) \in \mathscr{U}_j \cap \mathscr{U}_k$ and $|t|$ is sufficiently small. Let

$$\eta_j(z_j, t) = \sum_{v \geq 0} \eta_{j,v}(z_j) t^v$$

be the power series expansion of $\eta_j(z_j, t)$ and put

$$\eta_j^\mu(z_j, t) = \sum_{v=0}^{\mu} \eta_{j|v}(z_j) t^v.$$

Then (1) is equivalent to the system of congruences

$$(1)_\mu \quad \eta_j^\mu(f_{jk}(z_k, t), t) \equiv_{\mu} T_{jk}(z_k, t)\eta_k^\mu(z_k, t),$$

where \equiv denotes the identity mod. $t^{\mu+1}$. Obviously $\eta_j^0(z_j, t) = \eta_{j|0}(z_j) = \eta_j(z_j)$ satisfy $(1)_0$. Suppose that $\eta_j^{\mu-1}(z_j, t)$ satisfying $(1)_{\mu-1}$ are already determined. We define a vector-valued holomorphic function $\Gamma_{jk|_\mu}(z_k)$ defined on each $U_j \cap U_k$ by the congruence

$$\Gamma_{jk|\mu}(z_k)t^\mu \equiv \eta_j^{\mu-1}(f_{jk}(z_k, t), t) - T_{jk}(z_k, t)\eta_k^{\mu-1}(z_k, t).$$

Lemma 1. We have

$$\Gamma_{ik|\mu}(z_k) = \Gamma_{ij|\mu}(z_j) + T_{ij}(z_j)\Gamma_{jk|\mu}(z_k),$$

where $z_j = f_{jk}(z_k)$.

Proof. From the identity $T_{ik}(z_k, t) = T_{ij}(z_j, t)T_{jk}(z_k, t)$, we have

$$\Gamma_{jk|\mu}(z_k)t^\mu \equiv \eta_i^{\mu-1}(z_i, t) - T_{ij}(z_j, t)T_{jk}(z_k, t)\eta_k^{\mu-1}(z_k, t).$$

Using $T_{jk}(z_k, t)\eta_k^{\mu-1}(z_k, t) \equiv \eta_j^{\mu-1}(z_j, t) - \Gamma_{jk|\mu}(z_k)t^\mu$, we get

$$\begin{aligned} & T_{ij}(z_j, t)T_{jk}(z_k, t)\eta_k^{\mu-1}(z_k, t) \\ & \equiv T_{ij}(z_j, t)\eta_j^{\mu-1}(z_j, t) - T_{ij}(z_j, t)\Gamma_{jk|\mu}(z_k)t^\mu \\ & \equiv T_{ij}(z_j, t)\eta_j^{\mu-1}(z_j, t) - T_{ij}(z_j)\Gamma_{jk|\mu}(z_k)t^\mu. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Gamma_{ik|\mu}(z_k)t^\mu & \equiv \eta_i^{\mu-1}(z_i, t) - T_{ij}(z_j, t)\eta_j^{\mu-1}(z_j, t) + T_{ij}(z_j)\Gamma_{jk|\mu}(z_k)t^\mu \\ & \equiv \Gamma_{ij|\mu}(z_j)t^\mu + T_{ij}(z_j)\Gamma_{jk|\mu}(z_k)t^\mu. \end{aligned}$$

Consequently we have

$$\Gamma_{ik|\mu}(z_k) = \Gamma_{ij|\mu}(z_j) + T_{ij}(z_j)\Gamma_{jk|\mu}(z_k), \quad z_j = f_{jk}(z_k), \quad \text{Q.E.D.}$$

Now let $C^p(\mathfrak{U}, \Theta)$ denote the group of p -cochains of \mathfrak{U} with values in Θ , where $\mathfrak{U} = \{U_j\}$, $U_j = \mathcal{U}_j \cap V$. Moreover let $\delta^p: C^p(\mathfrak{U}, \Theta) \rightarrow C^{p+1}(\mathfrak{U}, \Theta)$ be the coboundary homomorphism and let $Z^p(\mathfrak{U}, \Theta) = \ker \delta^p$ be the subgroup of $C^p(\mathfrak{U}, \Theta)$ consisting of p -cocycles. Since \mathfrak{U} is a Stein covering of V , we have an isomorphism $H^p(V, \Theta) \xrightarrow{\sim} H^p(\mathfrak{U}, \Theta) = Z^p(\mathfrak{U}, \Theta)/\delta C^{p-1}(\mathfrak{U}, \Theta)$. The above lemma states that

$$\Gamma_\mu = \left\{ \sum_\alpha \Gamma_{jk|\mu}^\alpha(z_k) \frac{\partial}{\partial z_j^\alpha} \right\}$$

belongs to $Z^1(\mathfrak{U}, \Theta)$. Assume the existence of $\eta_j^\mu(z_j, t) = \eta_j^{\mu-1}(z_j, t) + \eta_{j|\mu}(z_j)t^\mu$ satisfying (1) _{μ} ; then we have $\Gamma_{jk|\mu}(z_k) \equiv_0 T_{jk}(z_k, t)\eta_{k|\mu}(z_k) - \eta_{j|k}(f_{jk}(z_k, t))$. Consequently we get

$$(2) \quad \Gamma_{jk|\mu}(z_k) = T_{jk}(z_k)\eta_{k|\mu}(z_k) - \eta_{j|k}(z_j),$$

where $z_j = f_{jk}(z_k)$. The identity (2) shows that the 1-cocycle Γ_μ falls off into a coboundary: $\Gamma_\mu = \delta \xi_\mu$, where ξ_μ is the 0-cochain given by

$$\xi_\mu = \left\{ \sum_{\alpha} \eta_{j|\mu}(z_j) \frac{\partial}{\partial z_j^\alpha} \right\}.$$

Conversely, if Γ_μ falls off into a coboundary, then there exist $\eta_j^\mu(z_j, t)$ satisfying the congruence (1) $_\mu$. We call Γ_μ the μ -th obstruction to extending η .

By definition, we have

$$\Gamma_{jk|1}(z_k)t \equiv \eta_j(f_{jk}(z_k, t)) - T_{jk}(z_k, t)\eta_k(z_k);$$

hence the first obstruction Γ_1 is determined uniquely by η . Thus, if the first obstruction does not vanish (i.e., does not fall off into a coboundary) then η is never extendible. But any 0-cocycle (i.e., holomorphic vector field) ξ may be added to $\xi_{\mu-1}$, and the higher obstructions have ambiguity to some extent. Thus, even if an obstruction Γ_μ does not vanish, by different choice of $\xi_{\mu-1}$ (or of $\eta_{j|\mu-1}(z_j)$), a modified obstruction Γ'_μ may vanish.

Finally, we note a special case of Theorem 2.3 in [3]:

Lemma 2. Let E denote the subspace of $H^0(V, \Theta)$ consisting of all the extendible elements. Then we have an isomorphism: $H^0(V, \Theta) \xrightarrow{\sim} E$, for $t \neq 0$.

§2. Calculation of lower obstructions

In this section, we deduce explicit forms of the first and second obstructions.

The usual Poisson bracket for vector fields induces a bilinear map: $[\cdot, \cdot]: C^p(\mathcal{U}, \Theta) \times C^q(\mathcal{U}, \Theta) \rightarrow C^{p+q}(\mathcal{U}, \Theta)$, which is defined by $[\alpha, \beta]_{i_0, \dots, i_{p+q}} = [\alpha_{i_0, \dots, i_p}, \beta_{i_{p+1}, \dots, i_{p+q}}]$, for $\alpha \in C^p(\mathcal{U}, \Theta)$ and $\beta \in C^q(\mathcal{U}, \Theta)$. We have

$$\delta[\alpha, \beta] = [\delta\alpha, \beta] + (-1)^p[\alpha, \delta\beta].$$

Hence we obtain a bilinear map $H^p(V, \Theta) \times H^q(V, \Theta) \rightarrow H^{p+q}(V, \Theta)$, which is also denoted by $[\cdot, \cdot]$. We have the following formulae:

$$[\phi, \psi] = (-1)^{pq+1}[\psi, \phi],$$

$$(-1)^{pr}[\phi, [\psi, \tau]] + (-1)^{qp}[\psi, [\tau, \phi]] + (-1)^{rq}[\tau, [\phi, \psi]] = 0,$$

for $\phi \in H^p(V, \Theta)$, $\psi \in H^q(V, \Theta)$ and $\tau \in H^r(V, \Theta)$.

Now we calculate the lower obstructions to extending a vector field η . Put $\chi_{jk}^\mu(z_k, t) = \eta_j^{\mu-1}(f_{jk}(z_k, t), t) - T_{jk}(z_k, t)\eta_k^{\mu-1}(z_k, t)$. Then, by definition, $\Gamma_{jk|\mu}(z_k)t^\mu \equiv \chi_{jk}^\mu(z_k, t)$. Let

$$\beta_{jk}(z_k, t) = \frac{\partial f_{jk}(z_k, t)}{\partial t}$$

and set

$$\theta_{jk} = \sum_{\alpha=1}^n \beta_{jk}^{\alpha}(z_k) \frac{\partial}{\partial z_j^{\alpha}} = \sum_{\alpha=1}^n \left(\frac{\partial f_{jk}^{\alpha}(z_k, t)}{\partial t} \right)_{t=0} \frac{\partial}{\partial z_j^{\alpha}}.$$

Then $\theta = \{\theta_{jk}\}$ is a 1-cocycle representing the image of the tangent vector d/dt at $0 \in D$ by the infinitesimal deformation map $\rho: T_0(D) \rightarrow H^1(V, \Theta)$

Convention. To avoid complication, we shall omit the symbol of summation, suffixes indicating the order of coordinates and arguments of functions if there is no fear of confusion. Thus $\frac{\partial \eta_j}{\partial z_j} \frac{\partial f_{jk}}{\partial t}$ stands for the vector

$$\left(\sum_{\alpha=1}^n \frac{\partial \eta_j^1(z_j)}{\partial z_j^{\alpha}} \frac{\partial f_{jk}^{\alpha}(z_k, t)}{\partial t}, \dots, \sum_{\alpha=1}^n \frac{\partial \eta_j^{\beta}(z_j)}{\partial z_j^{\alpha}} \frac{\partial f_{jk}^{\alpha}(z_k, t)}{\partial t}, \dots, \sum_{\alpha=1}^n \frac{\partial \eta_j^n(z_j)}{\partial z_j^{\alpha}} \frac{\partial f_{jk}^{\alpha}(z_k, t)}{\partial t} \right).$$

Proposition 1. The first obstruction Γ_1 to extending a holomorphic vector field η is given by

$$(3) \quad \Gamma_1 = [\theta, \eta].$$

Proof. We have

$$\begin{aligned} \Gamma_{jk|1} &= \left(\frac{\partial \chi_{jk}^1}{\partial t} \right)_{t=0} = \left(\frac{\partial \eta_j}{\partial z_j} \frac{\partial f_{jk}}{\partial t} - \frac{\partial T_{jk}}{\partial t} \eta_k \right)_{t=0} \\ &= \frac{\partial \eta_j}{\partial z_j} \beta_{jk} - \frac{\partial \beta_{jk}}{\partial z_k} \eta_k. \end{aligned}$$

Hence we get $\Gamma_{jk|1} \frac{\partial}{\partial z_j} = [\theta, \eta]_{jk}$, Q.E.D.

Next we assume that Γ_1 falls off into a coboundary:

$$\Gamma_1 = \delta \xi_1, \quad \xi_1 = \left\{ \eta_{j|1}(z_j) \frac{\partial}{\partial z_j} \right\}, \text{ i.e.,}$$

$$(4) \quad [\theta, \eta]_{jk} = \eta_{k|1} \frac{\partial}{\partial z_k} - \eta_{j|1} \frac{\partial}{\partial z_j}.$$

Proposition 2. The second obstruction Γ_2 to extending η is given by

$$(5) \quad \Gamma_2 = [\sigma + \varepsilon, \eta] + [\theta, \xi_1] - [\xi_1, \theta],$$

where σ and ε are 1-cochains defined, respectively, by

$$\sigma_{jk} = \gamma_{jk} \frac{\partial}{\partial z_j}, \quad \gamma_{jk}(z_k) = \left(\frac{\partial \beta_{jk}(z_k, t)}{\partial t} \right)_{t=0}$$

and

$$\varepsilon_{jk} = \beta_{kj} \frac{\partial \beta_{jk}}{\partial z_k} \frac{\partial}{\partial z_j}.$$

Proof. We have

$$\frac{\partial^2 \chi_{jk}^2}{\partial t^2} = \frac{\partial \eta_j^1}{\partial z_j} \frac{\partial f_{jk}}{\partial t} + \frac{\partial \eta_j^1}{\partial t} - \left(\frac{\partial T_{jk}}{\partial t} \eta_k^1 + T_{jk} \frac{\partial \eta_k^1}{\partial t} \right).$$

Hence we get

$$\begin{aligned} \frac{\partial^2 \chi_{jk}^2}{\partial t^2} &= \left(\frac{\partial^2 \eta_j^1}{\partial z_j^2} \frac{\partial f_{jk}}{\partial t} + 2 \frac{\partial^2 \eta_j^1}{\partial t \partial z_j} \right) \frac{\partial f_{jk}}{\partial t} + \frac{\partial \eta_j^1}{\partial z_j} \frac{\partial^2 f_{jk}}{\partial t^2} \\ &\quad - \left(\frac{\partial^2 T_{jk}}{\partial t^2} \eta_k^1 + 2 \frac{\partial T_{jk}}{\partial t} \frac{\partial \eta_k^1}{\partial t} \right). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \left(\frac{\partial^2 \chi_{jk}^2}{\partial t^2} \right)_{t=0} &= \frac{\partial^2 \eta_j}{\partial z_j^2} \beta_{jk}^2 + \frac{\partial \eta_j}{\partial z_j} \gamma_{jk} - \frac{\partial \gamma_{jk}}{\partial z_k} \eta_k \\ &\quad + 2 \left(\frac{\partial \eta_{j|1}}{\partial z_j} \beta_{jk} - \frac{\partial \beta_{jk}}{\partial z_k} \eta_{k|1} \right). \end{aligned}$$

On the other hand, using (4), we have

$$\begin{aligned} [\theta, \xi_1]_{jk} &= \left[\beta_{jk} \frac{\partial}{\partial z_j}, \eta_{k|1} \frac{\partial}{\partial z_j} \right] \\ &= \left[\beta_{jk} \frac{\partial}{\partial z_j}, \left(\beta_{jk} \frac{\partial \eta_j}{\partial z_j} - \eta_j \frac{\partial \beta_{jk}}{\partial z_j} \right) \frac{\partial}{\partial z_j} \right] + \left[\beta_{jk} \frac{\partial}{\partial z_j}, \eta_{j|1} \frac{\partial}{\partial z_j} \right] \\ &= \beta_{jk} \left(\frac{\partial \beta_{jk}}{\partial z_j} \frac{\partial \eta_j}{\partial z_j} + \beta_{jk} \frac{\partial^2 \eta_j}{\partial z_j^2} - \frac{\partial \eta_j}{\partial z_j} \frac{\partial \beta_{jk}}{\partial z_j} - \eta_j \frac{\partial^2 \beta_{jk}}{\partial z_j^2} \right) \frac{\partial}{\partial z_j} \\ &\quad + \left(\beta_{jk} \frac{\partial \eta_{j|1}}{\partial z_j} - \eta_{k|1} \frac{\partial \beta_{jk}}{\partial z_k} \right) \frac{\partial}{\partial z_j}. \end{aligned}$$

From this we get

$$\begin{aligned}
\Gamma_{jk|2} \frac{\partial}{\partial z_j} &= [\sigma, \eta]_{jk} + 2[\theta, \xi_1]_{jk} + \left(2\beta_{jk} \frac{\partial \eta_j}{\partial z_j} \frac{\partial \beta_{jk}}{\partial z_j} \right. \\
&\quad \left. + 2\beta_{jk} \eta_j \frac{\partial^2 \beta_{jk}}{\partial z_j^2} - 2\beta_{jk} \frac{\partial \beta_{jk}}{\partial z_j} \frac{\partial \eta_j}{\partial z_j} - \beta_{jk}^2 \frac{\partial^2 \eta_j}{\partial z_j^2} \right) \frac{\partial}{\partial z_j} \\
&= [\sigma, \eta]_{jk} + 2[\theta, \xi_1]_{jk} \\
&\quad + \left[\beta_{jk} \frac{\partial}{\partial z_j}, \left(\eta_j \frac{\partial \beta_{jk}}{\partial z_j} - \beta_{jk} \frac{\partial \eta_j}{\partial z_j} \right) \frac{\partial}{\partial z_j} \right] \\
&\quad - \left[\beta_{jk} \frac{\partial \beta_{jk}}{\partial z_j} \frac{\partial}{\partial z_j}, \eta_j \frac{\partial}{\partial z_j} \right] \\
&= [\sigma, \eta]_{jk} + 2[\theta, \xi_1]_{jk} + [\theta, [\theta, \eta]]_{jk} + [\varepsilon, \eta]_{jk}.
\end{aligned}$$

Taking account of the identity

$$\begin{aligned}
[\theta, [\theta, \eta]]_{jk} &= [\theta, \delta \xi_1]_{jk} \\
&= [\theta_{jk}, (\xi_1)_j] - [\theta_{jk}, (\xi_1)_k] \\
&= -[\xi_1, \theta]_{jk} - [\theta, \xi_1]_{jk},
\end{aligned}$$

we obtain the formula (5), Q.E.D.

Remark. We may add to ξ_1 any holomorphic vector field ξ . Hence the ambiguity of the second obstruction is $[\theta, \xi] - [\xi, \theta] = 2[\theta, \xi]$.

§3. Obstructions and jumping of complex structures

In this section, we examine relations between the obstructions to extending some vector field and jumping of complex structures, which are stated in parallel with Griffiths [3]. We also make some remarks on jumping of complex structures.

Let $\mathcal{V} \xrightarrow{\sim} D$ be a one-parameter family with coordinate transformations $z_j = f_{jk}(z_k, t)$. We put $\beta_{jk}(z_k, t) = [\partial f_{jk}(z_k, t)]/\partial t$, $T_{jk}(z_k, t) = [\partial f_{jk}(z_k, t)]/\partial z_k$, $\theta_{jk} = \beta_{jk}(z_k) \partial/\partial z_j$ and $\theta = \{\theta_{jk}\}$ as before. A 1-cocycle $\sigma = \{\sigma_{jk}\} \in Z^1(\mathcal{U}, \Theta)$ is said to be extendible in $\mathcal{V} \xrightarrow{\sim} D$ if there exists $\sigma(t) \in Z^1(\mathcal{U}_t, \Theta_t)$ for each $t \in D$, which depends holomorphically on t , where \mathcal{U}_t denotes the open covering $\{\mathcal{U}_j \cap V_t\}$ of V_t . More precisely, σ is extendible if we can find holomorphic functions $\gamma_{jk}(z_k, t)$ on $\mathcal{U}_j \cap \mathcal{U}_k$ with $\sigma_{jk} = \gamma_{jk}(z_k) \partial/\partial z_j$ and $\gamma_{ik}(z_k, t) = \gamma_{ij}(z_j, t) + T_{ij}(z_j, t) \gamma_{jk}(z_k, t)$. We put $\{\gamma_{jk}(z_k, t) \partial/\partial z_j\} = \sigma(t)$. An extension $\sigma(t)$ of σ is said to be a jump extension if there exists a vector-valued function $\xi_j(z_j, t)$ on each \mathcal{U}_j which is holomorphic in z_j and meromorphic in t with the pole only at $t = 0$, such that

$$(6) \quad \gamma_{jk}(z_k, t) = T_{jk}(z_k, t)\xi_k(z_k, t) - \xi_j(z_j, t) \text{ on } \mathcal{U}_j \cap \mathcal{U}_k.$$

The identity (6) shows that the 1-cycle $\sigma(t)$ falls off into a coboundary for $t \neq 0$. σ is said to be a jump cocycle if there exists a jump extension $\sigma(t)$ of σ . Lemmas 1.3–1.8, hence Theorem 1.1 in [3], are also valid in our case, which can be proved in an elementary way by considering the Laurent expansion of $\xi_j(z_j, t)$ in t . Let us quote one of them:

Lemma 3. An element σ of $Z^1(\mathcal{U}, \Theta)$ is a jump cocycle if and only if σ is an obstruction to extending some $\eta \in H^0(V, \Theta)$.

Now we consider the case in which $\theta(t) = \{\theta_{jk}(t)\}$, where $\theta_{jk}(t) = \beta_{jk}(z_k, t) \partial/\partial z_j$ itself is a jump extension of θ . Then $\rho_t: T_t(D) \rightarrow H^1(V_t, \Theta_t)$ is a zero map for $t \neq 0$. Moreover, $\dim H^1(V_t, \Theta_t)$ is independent of t for $t \neq 0$ ([3] Theorem 2.3). Hence the family $\mathcal{V} - V_0 \xrightarrow{\sim} D - \{0\}$ is locally trivial ([4] Theorems 6.2 and 18.2). Let $\mathcal{W} \xrightarrow{\sim} M$ denote a complex analytic family of complex structures. We say that the complex structure jumps at a point $0 \in M$ if $W_s = \pi^{-1}(s)$ is isomorphic to a fixed complex manifold W_1 for s near 0 and W_0 is not isomorphic to W_1 . Thus the complex structure in the above family $\mathcal{V} \xrightarrow{\sim} D$ jumps at 0.

Proposition. For any (local) one-parameter family $\mathcal{V} \xrightarrow{\sim} D$ in which the complex structure jumps at 0, we have a fiber preserving isomorphism $\mathcal{V} - V_0 \xrightarrow{\sim} V_1 \times (D - \{0\})$.

Proof. By a theorem of Fischer-Grauert [1], $\mathcal{V} - V_0 \xrightarrow{\sim} D - \{0\}$ is locally trivial, hence a fiber bundle with fiber a compact complex manifold V_1 and structure group $\text{Aut}(V_1)$, which is a complex Lie group. Now $\mathcal{V} \xrightarrow{\sim} D$, *a fortiori* $\mathcal{V} - V_0 \rightarrow D - \{0\}$ is topologically trivial. Since $D - \{0\}$ is a Stein manifold, by Oka's principle we see that $\mathcal{V} - V_0 \rightarrow D - \{0\}$ is also analytically trivial (see Grauert [2]).

Remark. If the dimension of the parameter space is greater than one, the above proposition is never valid in view of Hartog's continuation theorem.

Finally, we examine a special case in detail: Let $\mathcal{V} \xrightarrow{\sim} D$ be a one-parameter family of deformations of V . Assume that $\rho: T_0(D) \xrightarrow{\sim} H^1(V, \Theta)$ is an isomorphism. Let $z_j = f_{jk}(z_k, t)$ be coordinate transformations, and put

$$\beta_{jk}(z_k, t) = \frac{\partial f_{jk}(z_k, t)}{\partial t}, \quad T_{jk}(z_k, t) = \frac{\partial f_{jk}(z_k, t)}{\partial z_k}, \quad \theta_{jk} = \beta_{jk}(z_k) \frac{\partial}{\partial z_j} \text{ and } \theta = \{\theta_{jk}\}$$

as before. We consider the case in which there exists a vector field $\eta \in H^0(V, \Theta)$ such that

$$(7) \quad [\theta, \eta] \sim \theta,$$

where \sim denotes a cohomological relation. The vector field η generates a 1-parameter group $\{\phi_\tau\}$ of automorphisms ϕ_τ of V . From the completeness of the family $\mathcal{V} \rightarrow D$ (see [5]), we see that there exists an automorphism Φ_τ of \mathcal{V} which extends ϕ_τ . Hence we have a holomorphic vector field $\eta(z, t)$ on \mathcal{V} which extends the given η . We write

$$\eta(z, t) = \sum_{\alpha=1}^n \eta_j^\alpha(z_j, t) \frac{\partial}{\partial z_j^\alpha} + \xi_j(z_j, t) \frac{\partial}{\partial t} \quad \text{on } \mathcal{U}_j,$$

where $\eta_j^\alpha(z_j, t)$ and $\xi_j(z_j, t)$ are holomorphic functions on \mathcal{U}_j with the initial conditions $\eta_j(z_j, 0) = \eta_j(z_j)$ and $\xi_j(z_j, 0) = 0$. Taking into account the compatibility condition in the intersection $\mathcal{U}_j \cap \mathcal{U}_k$, we have

$$(8) \quad \eta_j(z_j, t) = T_{jk}(z_k, t) \eta_k(z_k, t) + \xi_k(z_k, t) \beta_{jk}(z_k, t),$$

$$(9) \quad \xi_j(z_j, t) = \xi_k(z_k, t).$$

The equation (9) shows that $\xi_j(z_j, t) = \xi_k(z_k, t) = \xi(t)$ depends only on t , since each fiber V_t is compact, and (8) is reduced to

$$(10) \quad \eta_j(z_j, t) = T_{jk}(z_k, t) \eta_k(z_k, t) + \xi(t) \beta_{jk}(z_k, t).$$

Let $\xi(t) = \sum_{\mu \geq 1} \xi_\mu t^\mu$ and $\eta_j(z_j, t) = \sum_{\mu \geq 0} \eta_{j|\mu}(z_j) t^\mu$ be the power series expansions. Differentiating (10) with respect to t , and putting $t = 0$, we obtain

$$\frac{\partial \eta_i}{\partial z_j} \beta_{jk} - \frac{\partial \beta_{jk}}{\partial z_k} \eta_k = T_{jk} \eta_{k|1} - \eta_{j|1} + \xi_1 \beta_{jk}.$$

In view of the assumption (7), we have $(1 - \xi_1)\theta \sim 0$. Hence we get $\xi_1 = 1$. From (10), we have

$$(11) \quad \beta_{jk}(z_k, t) = \frac{-1}{t + t^2 h(t)} (T_{jk}(z_k, t) \eta_k(z_k, t) - \eta_j(z_j, t)),$$

where $h(t)$ is a holomorphic function of t . The identity (11) implies that $\theta(t) = \left\{ \beta_{jk}(z_k, t) \frac{\partial}{\partial z_j} \right\}$ is a jump extension of θ , and hence the fiber manifold $\mathcal{V} - V_0 \rightarrow D - \{0\}$ is trivial. Moreover, take an arbitrary holomorphic function $g(t)$ of t ; it is easily seen that there exists a vector field η' on \mathcal{V} of the form

$$\eta'(z, t) = \eta_j'(z_j, t) \frac{\partial}{\partial z_j} + (t + t^2 g(t)) \frac{\partial}{\partial t} \quad \text{on } \mathcal{U}_j,$$

where $\eta'_j(z_j, t) = \frac{1 + tg(t)}{1 + th(t)} \eta_j(z_j, t)$. It is because of the ambiguity of $g(t)$ that the complex manifold V of this type does not admit a (locally) universal family (cf. [8] and [12]). An example of this type of family is given in §4, 1^o.

§4. Complete families of Hirzebruch manifolds

Let Σ_m denote the Hirzebruch manifold of degree m . Recall that Σ_m and its complete family are constructed as follows (cf. [6] p. 86, [7] pp. 44-49, and [9] p. 41 Example 2):

Take two copies U_1 and U_2 of \mathbb{C} with coordinates z_1 and z_2 , respectively. Let \mathbb{P}^1 be a projective line with an inhomogeneous coordinate ζ . Then Σ_m is described as $\Sigma_m = U_1 \times \mathbb{P}^1 \cup U_2 \times \mathbb{P}^1$, where $(z_1, \zeta_1) \in U_1 \times \mathbb{P}^1$ and $(z_2, \zeta_2) \in U_2 \times \mathbb{P}^1$ are identified if and only if

$$(12) \quad \begin{cases} \zeta_1 = z_2^m \zeta_2, \\ z_1 = 1/z_2. \end{cases}$$

Next, we define surfaces S_t parametrized by $t = (t_1, \dots, t_{m-1}) \in \mathbb{C}^{m-1}$ as follows: $S_t = U_1 \times \mathbb{P}^1 \cup U_2 \times \mathbb{P}^1$, where $(z_1, \zeta_1) \in U_1 \times \mathbb{P}^1$ and $(z_2, \zeta_2) \in U_2 \times \mathbb{P}^1$ are identified if and only if

$$(13) \quad \begin{cases} \zeta_1 = z_2^m \zeta_2 + \sum_{k=1}^{m-1} t_k z_2^k, \\ z_1 = 1/z_2. \end{cases}$$

Then $\mathcal{S} = \bigcup_{t \in \mathbb{C}^{m-1}} S_t$ forms a complex analytic family. It is not difficult to show that $\rho: T_0(\mathbb{C}^{m-1}) \xrightarrow{\sim} H^1(S_0, \Theta)$ is an isomorphism. Thus the family $\mathcal{S} \rightarrow \mathbb{C}^{m-1}$ is complete and effectively parametrized at $0 = (0, \dots, 0) \in \mathbb{C}^{m-1}$. We have $S_0 = \Sigma_m$. Moreover by a suitable change of coordinates of \mathbb{P}^1 , it is shown that $S_{(0, \dots, 0, t_k, 0, \dots, 0)} = \Sigma_{m-2k}$, where we let $\Sigma_n = \Sigma_{-n}$ if $n < 0$ (see [6]). As for the relation between \mathcal{S} and maximal families for certain rational singularities, see [10].

Now our objective is to determine S_t for any $t \in \mathbb{C}^{m-1}$. We begin by recalling the dimensions of the cohomology groups of Σ_m with coefficients in the sheaf Θ of germs of holomorphic vector fields;

$$(14) \quad \begin{cases} \dim H^0(\Sigma_m, \Theta) = \begin{cases} 6 & m = 0, \\ m + 5, & m > 0, \end{cases} \\ \dim H^1(\Sigma_m, \Theta) = \begin{cases} 0, & m = 0, \\ m - 1, & m > 0, \end{cases} \\ \dim H^2(\Sigma_m, \Theta) = 0 \end{cases}$$

Since Σ_0 and Σ_1 are rigid, we assume that $m \geq 2$ from now on. As any deformation of a Hirzebruch manifold is also a Hirzebruch manifold, if we can find $\dim H^0(S_t, \Theta_t)$, S_t is determined. For this purpose we apply the extension problem for vector fields. We choose a basis $\{\eta_1, \eta_2, \dots, \eta_{m+5}\}$ of $H^0(\Sigma_m, \Theta)$ as follows:

$$(15) \quad \left\{ \begin{array}{l} \eta_1 = \frac{\partial}{\partial z_1} = -z_2^2 \frac{\partial}{\partial z_2} + m z_2 \zeta_2 \frac{\partial}{\partial \zeta_2}, \\ \eta_2 = z_1 \frac{\partial}{\partial z_1} - \frac{m}{2} \zeta_1 \frac{\partial}{\partial \zeta_1} = -z_2 \frac{\partial}{\partial z_2} + \frac{m}{2} \zeta_2 \frac{\partial}{\partial \zeta_2}, \\ \eta_3 = z_1^2 \frac{\partial}{\partial z_1} - m z_1 \zeta_1 \frac{\partial}{\partial \zeta_1} = -\frac{\partial}{\partial z_2}, \\ \eta_4 = \zeta_1 \frac{\partial}{\partial \zeta_1} = \zeta_2 \frac{\partial}{\partial \zeta_2}, \\ \eta_{5+l} = z_1^l \zeta_1^2 \frac{\partial}{\partial \zeta_1} = z_2^{m-l} \zeta_2^2 \frac{\partial}{\partial \zeta_2}, \quad (0 \leq l \leq m). \end{array} \right.$$

The cohomology classes $\rho(\partial/\partial t_v)$, $v = 1, 2, \dots, m-1$, form a basis of $H^1(\Sigma_m, \Theta)$, and $\theta^{(v)} = \{\theta_{12}^{(v)}\}$, $\theta_{12}^{(v)} = z_2^v \frac{\partial}{\partial \zeta_1}$ is a 1-cocycle representing the cohomology class $\rho(\partial/\partial t_v)$. The first obstruction to extending η_x to the $\partial/\partial t_v$ direction is given as follows:

$$(16) \quad \left\{ \begin{array}{l} [\theta_{12}^{(v)}, \eta_1] = v z_2^{v+1} \frac{\partial}{\partial \zeta_1} = \begin{cases} v \theta_{12}^{(v+1)}, & 1 \leq v \leq m-2, \\ (m-1) \frac{\partial}{\partial \zeta_2}, & v = m-1, \end{cases} \\ [\theta_{12}^{(v)}, \eta_2] = \left(v - \frac{m}{2}\right) \theta_{12}^{(v)}, \\ [\theta_{12}^{(v)}, \eta_3] = \begin{cases} (1-m) \frac{\partial}{\partial \zeta_1}, & v = 1, \\ (v-m) \theta_{12}^{(v-1)}, & 2 \leq v \leq m-1, \end{cases} \\ [\theta_{12}^{(v)}, \eta_4] = \theta_{12}^{(v)}, \\ [\theta_{12}^{(v)}, \eta_{5+l}] = 2 z_1^{l-v} \zeta_1 \frac{\partial}{\partial \zeta_1} = 2 z_2^{v-l} \zeta_2 \frac{\partial}{\partial \zeta_2}, \quad 0 \leq l \leq m. \end{array} \right.$$

To investigate the extension problem for any vector field $\eta = \sum_{\alpha=1}^{m+5} a_\alpha \eta_\alpha$, $a_\alpha \in \mathbb{C}$ to any direction $\sum_{v=1}^{m-1} u_v \partial/\partial t_v$, we consider the one-parameter family $\mathcal{S} = \{S_t\}_{t \in \mathbb{C}}$ which is defined by $S_t = U_1 \times \mathbb{P}^1 \cup U_2 \times \mathbb{P}^1$, where $(z_1, \zeta_1) \in U_1 \times \mathbb{P}^1$ and $(z_2, \zeta_2) \in U_2 \times \mathbb{P}^1$ are identified if and only if

$$(17)_m \quad \begin{cases} \zeta_1 = z_2^m \zeta_2 + \sum_{v=1}^{m-1} u_v z_2^v t, \\ z_1 = 1/z_2. \end{cases}$$

In this family we have $\sigma = \varepsilon = 0$ (cf. Proposition 2). To calculate the second obstruction to extending η , we need the following quantities:

$$(18) \quad \begin{cases} \left[\theta_{12}^{(\mu)}, \frac{\partial}{\partial \zeta_2} \right] = 0, \\ \left[\theta_{12}^{(\mu)}, \frac{\partial}{\partial \zeta_1} \right] = 0, \\ \left[\theta_{12}^{(\mu)}, z_1^{l-v} \zeta_1 \frac{\partial}{\partial \zeta_1} \right] = z_1^{l-v-\mu} \frac{\partial}{\partial \zeta_1} = \begin{cases} \theta_{12}^{(v+\mu-l)}, & v \leq l < v+\mu, \\ z_1^{l-v-\mu} \frac{\partial}{\partial \zeta_1}, & v+\mu \leq l \leq m, \end{cases} \\ \left[\theta_{12}^{(\mu)}, z_2^{v-l} \zeta_2 \frac{\partial}{\partial \zeta_2} \right] = \begin{cases} \theta_{12}^{(v+\mu-l)}, & v+\mu-m+1 \leq l < v, \\ z_2^{v+\mu-l-m} \frac{\partial}{\partial \zeta_2}, & 0 \leq l \leq v+\mu-m. \end{cases} \end{cases}$$

In the fiber manifold we are considering, we have the following

Lemma. If the first and second obstructions vanish, then η is extendible.

Proof. Let us recall the notations for a general one-parameter family employed in the previous sections. Assume that the first and second obstructions to extending η in the family $(17)_m$ vanish. Then we have vector-valued holomorphic functions $\eta_{j|1}(z_j)$ and $\eta_{j|2}(z_j)$ which satisfy the congruence

$$(19) \quad \begin{aligned} & \frac{\partial f_{jk}(z_k, t)}{\partial z_k} (\eta_k(z_k) + \eta_{k|1}(z_k)t + \eta_{k|2}(z_k)t^2) \\ & - (\eta_j(f_{jk}(z_k, t)) + \eta_{j|1}(f_{jk}(z_k, t))t + \eta_{j|2}(f_{jk}(z_k, t))t^2) \equiv 0. \end{aligned}$$

In our case, any element of the matrix $\partial f_{jk}(z_k, t)/\partial z_k$ does not contain a power of t of order greater than 1. Moreover, the above calculation shows that $\frac{\partial f_{jk}(z_k, t)}{\partial z_k} \eta_{k|2}(z_k)$ and $\eta_{j|2}(f_{jk}(z_k, t))$ do not depend on t , and that $\eta_j(f_{jk}(z_k, t))$ or $\eta_{j|1}(f_{jk}(z_k, t))$ does not contain a power of t of order greater than 2 or 1, respectively. Hence the left side of the congruence (19) does not contain a power of t of order greater than 3. Consequently the third obstruction $\Gamma_{jk|3}$ is identically equal to zero, and we may assume $\eta_{j|3}(z_j) = 0$.

Hence we have $\Gamma_{jk|4} = 0$. Continuing this, we have $\eta_{j|\mu}(z_j) = 0$, for $\mu \geq 3$, Q.E.D.

Now the condition for the vanishing of the first and second obstruction is given by *homogeneous* polynomial equations in u_v 's. Hence if we put $S_{(t_1, \dots, t_{m-1})} = \Sigma_{m(t_1, \dots, t_{m-1})}$ for $(t_1, \dots, t_{m-1}) \in \mathbb{C}^{m-1}$, $m(t_1, \dots, t_{m-1})$ depends only on the direction of the vector (t_1, \dots, t_{m-1}) , i.e., on the ratio (t_1, \dots, t_{m-1}) , and the set $\{(t_1, \dots, t_{m-1}) \in \mathbb{C}^{m-1} \mid m(t_1, \dots, t_{m-1}) = \text{const.}\}$ forms a cone (minus the vertex) in the parameter space \mathbb{C}^{m-1} . From the above consideration, it is sufficient for our purpose if we consider the family of type (17)_m.

1°. The case in which $m = 2$. We have $\dim H^1(\Sigma_2, \Theta) = 1$. From the identity $[\theta^{(1)}, \eta_4] = \theta^{(1)}$, we see that this gives an example of §3.

2°. The case in which $m \geq 3$. We consider the extension problem for a vector field $\eta = \sum_{\alpha=1}^{m+5} a_\alpha \eta_\alpha$ in the family (17)_m. Put $\theta = \sum_{v=1}^{m-1} u_v \theta^{(v)}$. Then the first obstruction Γ_1 to extending η to the $\sum_{v=1}^{m-1} u_v \partial/\partial t_v$ direction is given by

$$(20) \quad \Gamma_1 = [\theta, \eta] = (Aa', \tau) - \delta \zeta_1.$$

In the above equation, A is the $(m-1) \times 4$ matrix whose (v, μ) component $a_{v\mu}$ is given by

$$(21) \quad a_{v\mu} = \begin{cases} (v-1)u_{v-1}, & \mu = 1, \\ \left(v - \frac{m}{2}\right)u_v, & \mu = 2, \\ -(m-v-1)u_{v+1}, & \mu = 3, \\ u_v, & \mu = 4, \end{cases} \quad 1 \leq v \leq m-1,$$

where we adopt a convention; $0u_0 = 0u_m = 0$. a' and τ are the column vectors ${}^t(a_1, \dots, a_4)$ and ${}^t(\theta^{(1)}, \dots, \theta^{(m-1)})$, respectively, and (Aa', τ) denotes the inner product of Aa' and τ . Moreover, ζ_1 is the 0-cochain given by

$$\begin{aligned} (\zeta_1)_1 &= (m-1)u_1a_3 \frac{\partial}{\partial \zeta_1} - 2 \sum_{l=v}^m \sum_{v=1}^{m-1} u_v a_{l+5} z_1^{l-v} \zeta_1 \frac{\partial}{\partial \zeta_1}, \\ (\zeta_1)_2 &= (m-1)u_{m-1}a_1 \frac{\partial}{\partial \zeta_2} + 2 \sum_{l=0}^{v-1} \sum_{v=1}^{m-1} u_v a_{l+5} z_2^{v-l} \zeta_2 \frac{\partial}{\partial \zeta_2}. \end{aligned}$$

Hence the condition for the vanishing of the first obstruction is given by

$$(22) \quad Aa' = 0.$$

Now we assume the equation (22) is satisfied. Letting $\xi = \sum_{\alpha=1}^{m+5} b_{\alpha} \eta_{\alpha}$ be an arbitrary vector field, the second obstruction Γ_2 to extending η is given by

$$(23) \quad \begin{aligned} \Gamma_2 &= [\theta, \xi_1] - [\xi_1, \theta] + 2[\theta, \xi] \\ &= 2(Ba'' + Ab, \tau) + \text{coboundary.} \end{aligned}$$

In the above equation, a'' and b denote the column vectors ${}^t(a_5, \dots, a_{m+5})$ and ${}^t(b_1, \dots, b_4)$, respectively, and B denotes the $(m-1) \times (m+1)$ matrix whose (v, λ) component $b_{v\lambda}$ is given by

$$(24) \quad b_{v\lambda} = \begin{cases} \sum_{\mu=\lambda}^{v-1} u_{\mu} u_{v-\mu+\lambda-1} & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \lambda < v, \\ 0 & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \lambda = v, \\ - \sum_{\mu=v}^{\lambda-1} u_{\mu} u_{v-\mu+\lambda-1} & \cdot \cdot \cdot \cdot \cdot \cdot \cdot v < \lambda < m+1, \\ - \sum_{\mu=v+1}^{m-1} u_{\mu} u_{m-\mu+v} & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \lambda = m+1. \end{cases}$$

Hence the second obstruction vanishes if and only if

$$(25) \quad Ba'' = -Ab, \text{ for some } b \in \mathbb{C}^4.$$

Note that B is expressed as $B = AC + B'$, where C denotes the $4 \times (m+1)$ matrix whose (μ, λ) component $c_{\mu\lambda}$ is given by

$$(26) \quad c_{\mu\lambda} = \begin{cases} u_1 & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \mu = \lambda = 1, \\ u_{\lambda-1} & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \mu = 2, 2 \leq \lambda \leq m, \\ \left(\frac{m}{2} - \lambda\right) u_{\lambda-1} & \cdot \cdot \cdot \mu = 4, 2 \leq \lambda \leq m, \\ u_{m-1} & \cdot \cdot \cdot \cdot \cdot \cdot u = 3, \lambda = m+1, \\ 0 & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \text{otherwise} \end{cases}$$

and B' denotes the $(m-1) \times (m+1)$ matrix whose (v, λ) component $b'_{v\lambda}$ is given by

$$(27) \quad b'_{v\lambda} = \begin{cases} \sum_{\mu=1}^{v-1} (u_{\mu}u_{v-\mu} - u_{v-1}u_1) & \dots \dots \dots 2 \leq v \leq m-1, \lambda = 1, \\ \sum_{\mu=\lambda}^{v-1} (u_{\mu}u_{v-\mu+\lambda-1} - u_v u_{\lambda-1}) & \dots \dots \dots \lambda < v, \\ 0 & \dots \dots \dots \lambda = v, \\ \sum_{\mu=v}^{\lambda-1} (u_v u_{\lambda-1} - u_{\mu} u_{v-\mu+\lambda-1}) & \dots \dots \dots v < \lambda < m+1, \\ \sum_{\mu=v+1}^{m-1} (u_{v+1} u_{m-1} - u_{\mu} u_{m-\mu+v}) & \dots \dots \dots \lambda = m+1. \end{cases}$$

From (22) and (25), the dimension d of the space of extendible vector fields is given by $d = \dim \ker A + \dim B^{-1}(\text{im } A) = \dim \ker A + \dim B'^{-1}(\text{im } A)$. We have $\dim B'^{-1}(\text{im } A) = \dim \ker B' + \dim(\text{im } A \cap \text{im } B')$. Hence we get the following

Theorem. *The dimension $d = d(u)$ of the space of extendible holomorphic vector fields on Σ_m in the family $(17)_m$ is given by*

$$d = m + 5 - \text{rank } S$$

where S denotes the $(m-1) \times (m+5)$ matrix (A, B') .

Thus, if we take a point $t = (t_1, \dots, t_{m-1})$ different from the origin in the parameter space \mathbb{C}^{m-1} , S_t is given by $S_t = \Sigma_{m(u)}$, where $m(u) = m - \text{rank } S$ and u is the ratio $(t_1 : \dots : t_{m-1})$. We can determine the subset of the parameter space \mathbb{C}^{m-1} in which the rank of S is equal to 2 as follows. First we calculate the 3×3 minors of A .

$$(28) \quad \Delta_1 = \begin{vmatrix} (v(0)-1)u_{v(0)-1}, & \left(v(0)-\frac{m}{2}\right)u_{v(0)}, & -(m-v(0)-1)u_{v(0)+1} \\ (v(1)-1)u_{v(1)-1}, & \left(v(1)-\frac{m}{2}\right)u_{v(1)}, & -(m-v(1)-1)u_{v(1)+1} \\ (v(2)-1)u_{v(2)-1}, & \left(v(2)-\frac{m}{2}\right)u_{v(2)}, & -(m-v(2)-1)u_{v(2)+1} \end{vmatrix}$$

$$= \sum_{i=0}^2 \left(v(i+1) - \frac{m}{2} \right) (-m - v(i)v(i+2) + 1) (u_{v(i+2)-1}u_{v(i)+1} - u_{v(i)-1}u_{v(i+2)+1})u_{v(i+1)}$$

$$\begin{aligned}
& + \sum_{i=0}^2 (v(i+1)v(i) - \frac{1}{2}v(i+2)m^2 + \frac{1}{2}v(i+2)m)(u_{v(i)+1}u_{v(i+1)} \\
& \quad - u_{v(i+1)+1}u_{v(i)})u_{v(i+2)-1} \\
& + \sum_{i=0}^2 (v(i+1)v(i+2)m - v(i+1)v(i+2) - \frac{1}{2}v(i)m)(u_{v(i)+1}u_{v(i+2)-1} \\
& \quad - u_{v(i+2)}u_{v(i+1)-1}) \times u_{v(i)+1},
\end{aligned}$$

where $v(i)$ are integers such that $1 \leq v(i) \leq m-1$ and we set $v(i+3) = v(i)$.

$$\begin{aligned}
(29) \quad \Delta_2 &= \begin{vmatrix} (v(0)-1)u_{v(0)-1}, & \left(v(0) - \frac{m}{2}\right)u_{v(0)}, & u_{v(0)} \\ (v(1)-1)u_{v(1)-1}, & \left(v(1) - \frac{m}{2}\right)u_{v(1)}, & u_{v(1)} \\ (v(2)-1)u_{v(2)-1}, & \left(v(2) - \frac{m}{2}\right)u_{v(2)}, & u_{v(2)} \end{vmatrix} \\
&= \sum_{i=0}^2 (v(i)v(i+1) + v(i+2))(u_{v(i)-1}u_{v(i+1)} - u_{v(i+1)-1}u_{v(i)})u_{v(i+2)}.
\end{aligned}$$

Especially, if $v(0) = 1$, (29) reduces to

$$(29)' \quad \Delta_2 = (v(1)-1)(v(2)-1)(u_{v(1)-1}u_{v(2)} - u_{v(2)-1}u_{v(1)})u_1.$$

$$\begin{aligned}
(30) \quad \Delta_3 &= \begin{vmatrix} (v(1)-1)u_{v(0)-1}, & -(m-v(0)-1)u_{v(0)+1}, & u_{v(0)} \\ (v(1)-1)u_{v(1)-1}, & -(m-v(1)-1)u_{v(1)+1}, & u_{v(1)} \\ (v(2)-1)u_{v(2)-1}, & -(m-v(2)-1)u_{v(2)+1}, & u_{v(2)} \end{vmatrix} \\
&= \sum_{i=0}^2 (-v(i+2)v(i+1) - m + 1)(u_{v(i+2)-1}u_{v(i+1)+1} \\
& \quad - u_{v(i+1)-1}u_{v(i+2)+1})u_{v(i)}
\end{aligned}$$

$$+ (m-1) \sum_{i=0}^2 v(i)(u_{v(i+2)+1}u_{v(i+1)} - u_{v(i+1)+1}u_{v(i+2)})u_{v(i)-1}$$

$$+ \sum_{i=0}^2 v(i)(u_{v(i+1)-1}u_{v(i+2)} - u_{v(i+2)-1}u_{v(i+1)})u_{v(i)+1}.$$

$$(31) \quad \Delta_4 = \begin{vmatrix} \left(v(0) - \frac{m}{2}\right)u_{v(0)}, & -(m-v(0)-1)u_{v(0)+1}, & u_{v(0)} \\ \left(v(1) - \frac{m}{2}\right)u_{v(1)}, & -(m-v(1)-1)u_{v(1)+1}, & u_{v(1)} \\ \left(v(2) - \frac{m}{2}\right)u_{v(2)}, & -(m-v(2)-1)u_{v(2)+1}, & u_{v(2)} \end{vmatrix}$$

$$\begin{aligned}
&= \sum_{i=0}^2 v(i+1)v(i+2)(u_{v(i+1)v(i+2)+1} - u_{v(i+2)}u_{v(i+1)+1})u_{v(i)} \\
&\quad + (m-1) \sum_{i=0}^2 v(i)(u_{v(i+2)+1}u_{v(i+1)} - u_{v(i+1)+1}u_{v(i+2)})u_{v(i)}.
\end{aligned}$$

Especially, if $v(2) = m-1$, (31) reduces to

$$(31)' \quad \Delta_4 = (m-1-v(0))(m-1-v(1))(u_{v(1)+1}u_{v(0)} - u_{v(1)}u_{v(0)+1})u_{m-1}.$$

Lemma. Assume $\Delta_3 = 0$.

(i) if $u_1 = \dots = u_v = 0$, then $u_{v+1} = 0$ for $1 \leq v \leq m-2$.

(ii) if $u_v = \dots = u_{m-1} = 0$, then $u_{v-1} = 0$ for $2 \leq v \leq m-1$.

Proof. (i) Setting $v(0) = v$, $v(1) = v+1$, and $v(2) = v+2$ in $\Delta_3 = 0$, we get $(m-v-1)(v+1)u_{v+1}^3 = 0$.

(ii) Setting $v(0) = v-2$, $v(1) = v-1$, and $v(2) = v$ in $\Delta_3 = 0$, we get $(v-1)(m+1-v)u_{v-1}^3 = 0$, Q.E.D.

From (28), (29), (30) and (31) together with the above lemma, we see that the rank of A is equal to 2 if and only if

$$(32) \quad u_v u_\mu - u_{v+1} u_{\mu-1} = 0, \quad 1 \leq v < \mu \leq m-1.$$

On the other hand, if (32) holds, we have $B' = 0$. Hence $\text{rank } S = 2$ if and only if (32) holds. Consequently, we have Σ_{m-2} on the cone defined by (32) minus the origin.

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